

OPPOSITE RELATION ON DUAL POLAR SPACES AND HALF-SPIN GRASSMANN SPACES

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ABSTRACT. We characterize the collinearity (adjacency) relation of half-spin Grassmann spaces in terms of the relation to be opposite in the corresponding collinearity graphs. Our characterization is closely related with results given [1] and [2]. Also we show that this characterization does not hold for dual polar spaces.

Dedicated to Prof. Helmut Karzel on the occasion of his 80-th birthday

1. INTRODUCTION

Let V be an n -dimensional vector space over a division ring. Denote by $\mathcal{G}_k(V)$ the *Grassmannians* consisting of all k -dimensional subspaces of V . Two distinct elements of $\mathcal{G}_k(V)$ are called *adjacent* if their intersection belongs to $\mathcal{G}_{k-1}(V)$ (the latter is equivalent to the fact that the sum of these subspaces is $(k+1)$ -dimensional). The cases when $k = 1, n-1$ are non-interesting, since any two distinct elements of $\mathcal{G}_k(V)$ are adjacent if $k = 1$ or $n-1$. From this moment we suppose that $1 < k < n-1$.

The *Grassmann graph* is the graph whose vertex set is $\mathcal{G}_k(V)$ and whose edges are pairs of adjacent elements. By well-known Chow's theorem [4], every automorphism of this graph is induced by a semilinear isomorphism of V to itself or to the dual vector space V^* , and the second possibility can be realized only in the case when $n = 2k$. The Grassmann graph is connected and the distance between $S, U \in \mathcal{G}_k(V)$ is equal to

$$k - \dim(S \cap U) = \dim(S + U) - k$$

(the distance between two vertexes of a connected graph is defined as the smallest number i such that there is a path of length i connecting the vertexes); in particular, the diameter of the Grassmann graph is finite. Two elements of $\mathcal{G}_k(V)$ are called *opposite* if the distance between them is equal to the diameter. It follows from Blunck–Havlicek's results [2] (see also [5]) that the adjacency relation can be characterized in terms of the relations to be opposite: distinct $S_1, S_2 \in \mathcal{G}_k(V)$ are adjacent if and only if there exists $S \in \mathcal{G}_k(V) \setminus \{S_1, S_2\}$ such that every element of $\mathcal{G}_k(V)$ opposite to S is opposite to S_1 or S_2 . In particular, this implies that every bijective transformation of $\mathcal{G}_k(V)$ preserving the relation to be opposite in both directions is an automorphism of the Grassmann graph.

In this note, we characterize the collinearity (adjacency) relation of half-spin Grassmann spaces in terms of the relation to be opposite in the corresponding

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collinearity graph. Also we give an example showing that this characterization does not hold for dual polar spaces.

Usual and polar Grassmann spaces (in particular, dual polar spaces and half-spin Grassmann spaces) are also known as the shadow spaces of buildings of type A_n , C_n , and D_n . So our considerations can be also motivated by Abramenko–Van Maldeghem’s result [1] concerning the adjacency and opposite relations in the chamber sets of spherical buildings.

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2. DEFINITIONS AND RESULTS

Recall that a *partial linear space* is a pair $\Pi = (P, \mathcal{L})$, where P is a set of points and \mathcal{L} is a family of line such that each line contains at least two points, every point is on a certain line, and for any distinct points there is at most one line containing them (points connected by a line are called *collinear*). For every partial linear space $\Pi = (P, \mathcal{L})$ there is the associated *collinearity graph* whose point set is P and whose vertexes are pairs of collinear points.

Following [3] we define a *polar space* of finite rank as a partial linear space $\Pi = (P, \mathcal{L})$ satisfying the axioms:

- (1) on every line there are at least 3 points,
- (2) a point is collinear with all points of a line or with precisely one point of a line (Buekenhout–Shult’s axiom),
- (3) for each $p \in P$ there is a point non-collinear with p (our polar space is non-degenerate),
- (4) every flag consisting of singular subspaces is finite.

Then all maximal singular subspaces are projective spaces of same finite dimension n , and the number $n + 1$ is known as the *rank* of our polar space. The collinearity relation will be denoted by \perp : we write $p \perp q$ if $p, q \in P$ are collinear and $p \not\perp q$ otherwise. Similarly, $X \perp Y$ means that every point of X is collinear with all points of Y . We denote by X^\perp the set of all points $p \in P$ satisfying $p \perp X$.

A subset $F = \{p_1 \dots p_{2n+2}\}$ (recall that the rank of Π is equal to $n + 1$) is called a *frame* of Π if for every $p_i \in F$ there is precisely one point $p_j \in F$, $j \neq i$ such that $p_i \not\perp p_j$. In what follows we will use the following well-know fact: for every two singular subspaces there is a frame whose points span both these subspaces.

For every natural $n \geq 2$ there are precisely the following two types of rank n polar spaces:

- (C_n) every $(n - 2)$ -dimensional singular subspace is contained in at least three maximal singular subspaces,
- (D_n) every $(n - 2)$ -dimensional singular subspace is contained in precisely two maximal singular subspaces,

we say that a rank n polar space is of type C_n or D_n if the corresponding case is realized.

Let $\Pi = (P, \mathcal{L})$ be a polar space of rank $n \geq 3$. Denote by $\mathcal{G}_k(\Pi)$ the Grassmanian consisting of all k -dimensional singular subspaces. A subset of $\mathcal{G}_{n-1}(\Pi)$ is called a *line* if it consists of all maximal singular subspaces containing certain $M \in \mathcal{G}_{n-2}(\Pi)$. The *dual polar space* $\mathfrak{G}_{n-1}(\Pi)$ is the partial linear space whose points are elements

of $\mathcal{G}_{n-1}(\Pi)$ and whose lines are defined above. If our polar space is of type D_n the dual polar space is trivial: every line consists of precisely two points. We say that two elements $S, U \in \mathcal{G}_{n-1}(\Pi)$ are *opposite* and write $S \text{ op } U$ if the distance between them in the collinearity graph of $\mathfrak{G}_{n-1}(\Pi)$ is maximal; this is equivalent to the fact that S and U are disjoint.

Now suppose that our polar space is of type D_n , $n \geq 3$. Then the Grassmannian $\mathcal{G}_{n-1}(\Pi)$ can be presented as the sum of two disjoint subsets

$$\mathcal{O}_\delta(\Pi), \quad \delta \in \{+, -\}$$

such that the distance

$$d(S, U) = n - 1 - \dim(S \cap U)$$

(in the collinearity graph of $\mathfrak{G}_{n-1}(\Pi)$) is even if S, U belongs to the same $\mathcal{O}_\delta(\Pi)$ and odd otherwise. These subsets are known as the *half-spin Grassmannians* of Π . A subset of $\mathcal{O}_\delta(\Pi)$ is called a *line* if it consists of all elements of $\mathcal{O}_\delta(\Pi)$ containing certain $M \in \mathcal{G}_{n-3}(\Pi)$. We get a partial linear space which will denoted by $\mathfrak{D}_\delta(\Pi)$. In the case when $n = 3$, any two distinct elements of $\mathcal{O}_\delta(\Pi)$ are connected by a line (their intersection is a single point) and $\mathfrak{D}_\delta(\Pi)$ is a 3-dimensional projective space. If $n = 4$ then $\mathfrak{D}_\delta(\Pi)$ is a polar space of type D_4 .

As above, two elements $S, U \in \mathcal{O}_\delta(\Pi)$ are said to be *opposite*, $S \text{ op } U$, if the distance between them in the collinearity graph of $\mathfrak{D}_\delta(\Pi)$ is maximal. If n is even then this is equivalent to the fact that S and U are disjoint. In the case when n is odd, we have $S \text{ op } U$ if and only if the intersection of S and U is a single point.

Theorem 1. *If Π is of type D_n , $n \geq 4$ then the following conditions are equivalent*

- (1) $S_1, S_2 \in \mathcal{O}_\delta(\Pi)$ are collinear points of $\mathfrak{D}_\delta(\Pi)$,
- (2) there exists $S \in \mathcal{O}_\delta(\Pi) \setminus \{S_1, S_2\}$ such that $U \text{ op } S$ implies that $U \text{ op } S_1$ or $U \text{ op } S_2$.

Corollary. *Every bijective transformation of $\mathcal{O}_\delta(\Pi)$ preserving the relation to be opposite is a collineation of $\mathfrak{D}_\delta(\Pi)$.*

In Section 4 we show that Theorem 1 does not hold for dual polar spaces associated with sesquilinear forms.

3. PROOF OF THEOREM 1

In this proof we will distinguish the following two cases:

- (I) n is even, then $S \text{ op } U$ is equivalent to the fact that $S \cap U = \emptyset$,
- (II) n is odd, then $S \text{ op } U$ if and only if $S \cap U$ is a single point.

(1) \implies (2). Show that every point $S \neq S_1, S_2$ on the line joining S_1 with S_2 is as required (this line consists of all elements of $\mathcal{O}_\delta(\Pi)$ containing $S_1 \cap S_2$). Suppose that $U \text{ op } S$, but U is not opposite to both S_1 and S_2 .

Case (I). In this case, U intersects S_1 and S_2 by subspaces whose dimensions are not less than 1. We take lines $L_i \subset U \cap S_i$, $i = 1, 2$. These lines do not intersect $S_1 \cap S_2$. Hence S_i is spanned by $S_1 \cap S_2$ and the line L_i . The latter means that $L_1 \not\subset L_2$ which contradict the fact that our lines are contained in U .

Case (II). According our assumption, the dimensions of $U \cap S_1$ and $U \cap S_2$ are not less than 2. Let P_i be a plane contained in $U \cap S_i$, $i = 1, 2$. The planes P_1, P_2 both have a non-empty intersections with $S_1 \cap S_2$ (because $S_1 \cap S_2$ is $(n-3)$ -dimensional). Since $U \text{ op } S$, these intersections both are 0-dimensional. This implies the existence

of lines $L_i \subset P_i$, $i = 1, 2$ disjoint with $S_1 \cap S_2$. As in the previous case, $L_1 \not\subset L_2$ which is impossible.

Therefore, in the both cases we have $U \text{ op } S_i$ for at least one $i \in \{1, 2\}$.

(2) \implies (1). We prove this implication in several steps.

First we establish that *for every distinct collinear points $p_i \in S_i$ ($i = 1, 2$) the line p_1p_2 intersects S .*

Proof in the case (I). Suppose that the line p_1p_2 is disjoint with S . There exists a maximal singular subspace U containing p_1p_2 and opposite to S (we can take any frame of Π whose points span S and the line p_1p_2 , the maximal singular subspace spanned by points of the frame and opposite to S is as required). By our hypothesis, U is opposite to S_1 or S_2 ; this means that p_1 or p_2 is not in U which contradicts to the fact that line p_1p_2 is in U . \square

Proof in the case (II). The intersection of S_1 and S_2 is not empty. If the line p_1p_2 intersects $S_1 \cap S_2$ then $p_2 \in S_1$ and $p_1 \in S_2$; thus there are the following two possibilities:

$$p_1p_2 \subset S_1 \cap S_2 \text{ or } p_1p_2 \cap (S_1 \cap S_2) = \emptyset.$$

In each of these cases, we can choose a point $p \in S_1 \cap S_2$ which is not on the line p_1p_2 (in the first case, the dimension of $S_1 \cap S_2$ is not less than 2).

Consider the plane P spanned by p, p_1, p_2 . Assume that P intersects S precisely by a certain point. Using the existence of a frame whose points span P and S , we construct a maximal singular subspace U opposite to S and containing P . Then U is opposite to at last one of S_1, S_2 which contradicts the fact that the lines pp_1 and pp_2 are contained in U .

Now suppose that $P \cap S = \emptyset$. We choose a point q from $S \cap P^\perp$ (this is possible since n is not less than 4) and extend $\overline{P \cup \{q\}}$ to a maximal singular subspace U opposite to S (using a frame whose points span $\overline{P \cup \{q\}}$ and S). The dimension of each $S_i \cap U$ is not less than 1 which is impossible.

Therefore, $\dim(P \cap S) \geq 1$ and $P \cap S$ contains a line; this line intersects p_1p_2 (since P is a plane). \square

Our next step is the equalities

$$\dim(S \cap S_i) = n - 3, \quad i = 1, 2.$$

Proof. Let us take a point $p \in S_2 \setminus S$. Then $S_1 \cap p^\perp$ is a hyperplane of S_1 or it coincides with S_1 . Consider a line $L \subset S_1 \cap p^\perp$. Let u, v be distinct points on this line. The lines up and vp intersect S by points u' and v' , respectively. Since $p \notin S$, we have $p \neq u', v'$ and the points u', v' are distinct. The lines L and $u'v'$ both are contained in the plane $\overline{L \cup \{p\}}$, thus they have a non-empty intersection. The inclusion $u'v' \subset S$ guarantees that L intersects S .

So, every line $L \subset S_1 \cap p^\perp$ has a non-empty intersection with S . Thus S intersects $S_1 \cap p^\perp$ at least by a hyperplane. The dimension of $S_1 \cap p^\perp$ is not less than $n - 2$ and we get

$$\dim(S \cap S_1 \cap p^\perp) \geq n - 3$$

which implies that $S \cap S_1$ is $(n - 3)$ -dimensional. Similarly, we show that the dimension of $S \cap S_2$ is equal to $n - 3$. \square

Now establish the equality $\dim S_1 \cap S_2 = n - 3$ which completes our proof.

Proof. Define a

$$U := (S \cap S_1) \cap (S \cap S_2).$$

Since $S \cap S_1$ and $S \cap S_2$ are $(n-3)$ -dimensional subspaces of S , one of the following possibilities is realized:

- (1) $S \cap S_1 = S \cap S_2$ and U is $(n-3)$ -dimensional,
- (2) $\dim U = n-4$,
- (3) $\dim U = n-5$.

Since U is contained in $S_1 \cap S_2$, the dimension of $S_1 \cap S_2$ is equal to $n-3$ in the first and second cases.

Let U be an $(n-5)$ -dimensional subspace. If U does not coincide with $S_1 \cap S_2$ then $S_1 \cap S_2$ is $(n-3)$ -dimensional.

Now suppose that $U = S_1 \cap S_2$. We take any line $L \subset S_1 \setminus S$ and consider the singular subspace $L^\perp \cap S_2$; its dimension is not less than $n-3$. Moreover, this subspace does not contain $S \cap S_2$. Indeed, S is spanned by $S \cap S_1$ and $S \cap S_2$, and the inclusion

$$S \cap S_2 \subset L^\perp \cap S_2$$

implies that $L \perp S$; the latter is impossible, since S is a maximal singular subspace and $L \not\subset S$.

Therefore, there is a point $p \in S_2 \setminus S$ satisfying $p \perp L$. As above, we show that the intersection of S with the plane $\overline{L \cup \{p\}}$ contains a line. This line intersects L which contradicts $L \subset S_1 \setminus S$. This means that the third case can not be realized. \square

4. EXAMPLE

Let V be a left vector space over a division ring R and $\Omega : V \times V \rightarrow R$ be a non-degenerate reflexive sesquilinear form of Witt index $n \geq 3$. We write $\Pi = (P, \mathcal{L})$ for the associated polar space (P and \mathcal{L} are the sets of 1-dimensional and 2-dimensional totally isotropic subspaces, respectively) and suppose that it is of type C_n . Every element of $\mathcal{G}_{n-1}(\Pi)$ can be obtained from a certain maximal singular subspace of the form Ω .

We assert that the following conditions are not equivalent

- (1) $S_1, S_2 \in \mathcal{G}_{n-1}(\Pi)$ are collinear points of $\mathfrak{G}_{n-1}(\Pi)$,
- (2) there exists $S \in \mathcal{G}_{n-1}(\Pi) \setminus \{S_1, S_2\}$ such that $U \text{ op } S$ implies that $U \text{ op } S_1$ or $U \text{ op } S_2$.

It is not difficult to see that (1) implies (2) (any $S \in \mathcal{G}_{n-1}(\Pi) \setminus \{S_1, S_2\}$ belonging to the line joining S_1 with S_2 is as required). Now we show that (2) does not imply (1).

Let $p_1, \dots, p_n, q_1, \dots, q_n$ be a frame of Π such that $p_i \not\perp q_i$ for each i . For some vectors $x_1, \dots, x_n, y_1, \dots, y_n \in V$ we have

$$p_i = \langle x_i \rangle, \quad q_i = \langle y_i \rangle \quad \text{and} \quad \Omega(x_i, y_i) = 1.$$

The maximal singular subspaces of Π associated with the maximal totally isotropic subspaces

$$\begin{aligned} &\langle x_1, x_2, x_3, \dots, x_n \rangle, \\ &\langle y_1, y_2, x_3, \dots, x_n \rangle, \\ &\langle x_1 + y_2, x_2 - y_1, x_3, \dots, x_n \rangle \end{aligned}$$

will be denoted by S_1, S_2 , and S (respectively). Their intersection is the $(n-3)$ -dimensional singular subspace N associated with $\langle x_3, \dots, x_n \rangle$.

Now consider the line L joining $\langle x_1 + y_2 \rangle$ with $\langle x_2 - y_1 \rangle$. Every point on this line is of type

$$(1) \quad \langle (x_1 + y_2) + t(x_2 - y_1) \rangle, \quad t \in R$$

If $p \in p_1 p_2 \setminus \{p_1 p_2\}$ and $q \in q_1 q_2 \setminus \{q_1 q_2\}$ are collinear then

$$p = \langle x_1 + ax_2 \rangle \quad \text{and} \quad q = \langle y_2 - ay_1 \rangle$$

for a certain scalar $a \in R$; every point on the line pq is of type

$$(2) \quad \langle x_1 + ax_2 + s(y_2 - ay_1) \rangle, \quad s \in R$$

The lines L and pq have a non-empty intersection (because (1) coincides with (2) if $t = a$ and $s = 1$).

Similarly, we establish that for any two collinear points $p \in S_1 \setminus N$ and $q \in S_1 \setminus N$ the line pq intersects $S \setminus N$. Therefore, if $U \in \mathcal{G}_{n-1}(\Pi)$ is opposite to S then it is opposite to S_1 or S_2 . However, S_1 and S_2 are not collinear.

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